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Mathematics

Equilibria of Point Charges in Convex Domains

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ABSTRACT. We discuss equilibrium configurations of Coulomb potential of point charges in convex domains of the plane and three-dimensional Euclidean space. For a triple of points, we give an analytic criterion of the existence of point charges for which the given triple is an equilibrium configuration. Using this criterion, rather comprehensive results are obtained for three charges in the circle and ellipse. Several related problems and possible generalizations are also indicated. *© 2015 Bull. Georg. Natl. Acad. Sci.*

Key words: *point charge, Coulomb potential, critical point, equilibrium configuration, tangent vector, concurrent lines.*

1. The behaviour of point charges with Coulomb interaction confined to certain subsets (conductors) in Euclidean space is a classical topic of electrostatics and potential theory [1]. In particular, equilibrium distributions of point charges for various classes of subsets received considerable attention (see, e.g., [2, 3]). Mathematically, this reduces to the study of critical points of Coulomb potential restricted to the subset considered. Calculating the coordinates and establishing the topological type of those critical points is recognized as an important and difficult problem. Suffice it to mention the longstanding Maxwell conjecture on the number of equilibria of a finite system of point charges in $\mathbf{R}^3$ [3].

Recently, a seemingly new direction of research within this circle of ideas was suggested in [4, 5]. Several later developments in this direction are presented in [6, 7]. The main novelty of the approach developed in [4, 5] is that it focuses on the following problem which may be thought of as the *inverse problem of electrostatics* (IPES). To describe it more precisely, let us denote by $E_{Q/P}$ the Coulomb energy [1] of a collection of n point charges Q placed at n-tuple of points P. Each pair (Q,P) is called a *configuration of charges*. To refer to this setting we also use the notation Q/P.

(IPES) Given a finite configuration $P = (P_1, \dots, P_n)$ of points in a fixed subset X, does there exist a collection of non-zero real numbers $Q = (q_1, \dots, q_n)$, interpreted as values of point charges such that if they are placed at the points P_i then the given configuration is a critical point of Coulomb energy $E_{Q/P}$ restricted to X^n?

If such charges exist they will be called *stationary charges for (configuration) P* in X. Any configuration having stationary charges will be called a *Coulomb equilibrium* in X. If P is a local minimum of $E_{Q/P}$ then it will be called a *stable equilibrium of Q/P*. In such a case, a collection of stationary charges Q will be called a *stabilizer* of P.

Obviously, all these concepts and definitions substantially depend on the chosen subset X. Some choices of X are related to classical problems of electrostatics and go under various names. For example, if X is a non-intersecting smooth closed curve it can be considered as a mathematical analog (model) of a *thin conducting wire*. If X is a domain in the plane this corresponds to the model of *conducting plate,* and so on.

Many natural and interesting problems arise if one takes conductor X to be a smooth submanifold of Euclidean space. In other words, the constraints imposed on the positions of charges have the form of equations. A natural class of such problems was discussed in [5] in the setting of *equilibria of quadratically constrained point charges* which generalizes the case of vertex-charged polygonal linkages studied in [6] and the case of point charges on the circle or ellipse studied in [5, 7] (cf. also [8]). The IPES becomes very complicated if the number of charges is big and we do not possess reasonable results in the general case. However, it seems to deserve attention since the results and considerations in [6, 7] show that non-trivial and interesting issues arise already for three point charges on a closed curve and four point charges at the vertices of a quadrilateral linkage.

In the present paper, we make a few next steps in the framework of the same general paradigm. Namely, we study the equilibria of three point charges in a bounded convex smooth domain X in $\mathbf{R}^2$ or $\mathbf{R}^3$. In other words, we now deal with one constraint in the form of inequality. Although some of our considerations are valid for general convex domains, here we discuss only the cases of an ellipse and unit ball in three-dimensional Euclidean space, which can be treated in a lucid and concise way.

Our results are closely related to and rely on certain results from [5] and [7] concerned with the equilibria of point charges on a smooth contour K which is the boundary of a convex domain D. At the same time it should be noted that, as it will become quite clear in the sequel, there is an essential difference between the two settings. Although many of Coulomb equilibria ***in domain*** D coincide with the Coulomb equilibria ***in the boundary*** $K = dD$ there may also exist Coulomb equilibria which contain some points inside D. The easiest example of the latter possibility is given by a centrally symmetric configuration consisting of three equal charges placed at the center and two antipodal points of the unit circle.

There is an essential difference between these two types of Coulomb equilibria since the equilibria at the boundary can be considered as stable in a certain sense, while the above example seems to be unstable, which is typical for equilibria containing interior points. For boundary equilibria, a relevant notion of stability of Coulomb equilibrium arises if one requires that the Coulomb potential of stationary charges have a *non-degenerate* local minimum at this configuration. In the sequel such configurations will be called *Coulomb stable.* For example, it is easy to verify that the three equal charges at the vertices of regular triangle in S gives a stable configuration. We found out that this is a typical phenomenon for equilibria at the boundary, which yields explication of some results from [5, 7].

A straightforward way to rigorously treat the stability issue in situations similar to the above centrally symmetric configuration is to use a version of Morse theory valid for manifolds with corners which is beyond the main aims of this paper. So we discuss stability only for the Coulomb equilibria having all points at the boundary.

We also wish to mention another essential difference of our results on Coulomb equilibria from the previously known ones. Most of the previously known results refer to the case of equal charges [1, 2] while a crucial point for us is to consider the equilibrium configurations of not necessarily equal charges.

We begin by discussing in some detail the case of three charges in the unit disc D which may be considered as a paradigm for our further considerations. Our first main result (Theorem 1) gives a complete description of Coulomb stable configurations in the unit disc. This result relies on the study of three point charges on the unit circle $S = dD$ performed in [5]. We also present some corollaries of Theorem 1 concerned, in particular, with the case of four charges.

Next, we consider three charges in an arbitrary ellipse E and show that there exists a complete geometric description of Coulomb equilibria in this case too (Theorem 2). It seems worth noting that equilibria of three charges on an ellipse were numerically studied in [8]. However, [8] gives no general criterion of stability and, in fact, contains some erroneous conclusions. Our Theorem 2 yields, in particular, an explication of certain results of [8]. Finally, we obtain a simple description of Coulomb equilibria in the three-dimensional unit ball (Theorem 3).

2. We begin by recalling some features of the setting of [5] in the form adjusted to our purposes. For us it is reasonable to consider charges of the same sign. This permits to avoid some problems connected with the possible collisions and annihilation of charges which are beyond the accepted mathematical setting. So, throughout the paper we assume that all charges are positive. Recall that the *Coulomb (electrostatic) energy* of a system of point charges $Q = (q_1, \dots, q_n)$ placed at the points $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$, ..., $P_n = (x_n, y_n)$ in a subset X of the plane or three-dimensional (3d) Euclidean space (up to a constant multiple which we omit as irrelevant for our considerations) is defined as

$$E_{Q/P} = \sum_{i<j} \frac{q_i q_j}{d_{ij}^3},$$

where d_{ij} is the distance between points P_i and P_j. As is well known, the resultant force acting on q_i in position P_i is equal to

$$F_i = \sum F_{ji} = \sum \frac{q_i q_j}{d_{ij}^3}(p_i - p_j),$$

where p_i denotes the radius-vector of point P_i and $F_{ji} = \frac{q_i q_j}{d_{ij}^3}(p_i - p_j)$ is the electrostatic force (under our assumption it is a repelling force) acting on q_i at P_i due to its interaction with q_j at P_j. If charges $Q = (q_1, \dots, q_n)$ placed at $P_1, \dots, P_n$ stay in rest in X then we say that configuration $P = (P_1, \dots, P_n)$ is a *Coulomb equilibrium* for the collection of charges Q. In our setting this is equivalent to requiring that P is a critical point of $E_{Q/P}$ considered as function on X^n while the collection of charges Q remains fixed. In such a case we say that collection of charges Q is *stationary* for P. One of our main aims is to investigate and geometrically characterize those configurations P for which there exists a collection of stationary charges Q. Such configurations will be called *Coulomb equilibria* in X. As was mentioned, we will only consider the cases where X is a smooth convex domain in the plane or $\mathbf{R}^3$.

Consider first the case of three positive charges in the unit disc D in the plane. We begin with two general remarks used in the proof.

Remark 1. It is easy to verify that if a triple $P \subset D$ is a Coulomb equilibrium in D for a system of positive charges, then at least two of its points should lie on S. Indeed, otherwise it is geometrically obvious that one can find a small displacement (expanding deformation) of P in D such that all pairwise distances do not decrease and one of them is strictly bigger than in P, which implies that P is not a Coulomb equilibrium. However, the aforementioned example of centrally symmetric triple (two antipodes and the origin) with equal charges shows that one of the equilibrium points may lie inside D.

Remark 2. As was observed in [5], any finite configuration of points in the circle S which is a Coulomb equilibrium in S has the following geometric property: there exists no open semi-circle containing all these points. For brevity, we say that such a collection of points is *balanced* in S.

We are now ready to formulate and prove the first main result of this note. Its proof follows by reducing to the case of unit circle and some straightforward calculations. We present it in some detail since a similar result in the case of circle was given in [5] without proof for the reason of space. Recall that we consider only non-zero values of charges.

Theorem 1. Any triple of points P in the unit disc D which is a Coulomb equilibrium of three positive charges placed at these points, belongs to one of the two types: (1) (condiametric) two of the points are antipodal and the third point lies on the corresponding diameter, or (2) (marginal) it is a balanced triple of points in S. In all cases, except the centrally symmetric triple and vertices of a regular triangle in the unit circle, the stationary charges are uniquely defined up to a non-zero multiple. Any marginal Coulomb triple P is stable, in other words, configuration P is a local minimum of $E_{Q/P}$ for any collection Q of positive stationary charges.

Proof of Theorem 1. By Remark 1 we can assume that no more than one point lies inside D. Let us first show that if P contains an interior point then it is an antipodal triple. Notice that the two boundary points should be antipodal. Indeed, otherwise the third (interior) point can never stay in rest since the two electrostatic forces acting on it can only be balanced if their directions are opposite. For the same reason the third point should lie on the diameter connecting the two antipodes. So it is indeed sufficient to consider only two possibilities indicated in the theorem.

In the first (condiametric) case it is well-known that such three points have stationary charges and it is easy to verify that there always exists a triple of positive stabilizing charges. In fact, it is also well-known and easy to verify that such points can be also stabilized with respect to Hook law and other central forces.

In the second (boundary) case we can essentially refer to the description of Coulomb triples on the circle given in [5]. However, we present the proof in some detail since the mentioned description was given in [5] without proof and discussion of stability issue.

So, according to the general strategy of [5] we denote by q_1, q_2, q_3 the sought stationary charges and aim at obtaining a system of linear equations for q_1, q_2, q_3. To this end we write down the analytic expression of the fact that each point is in equilibrium for this system of charges. By Lagrange rule, at an equilibrium the resultant force should be orthogonal to the tangent vector to S at each point P_i. This gives three relations $(F_i, T_i) = 0$, $i=1,2,3$ which are obviously quadratic with respect to charges. Notice that each term of the i-th equation contains the charge q_i as a multiple. Since we search for non-zero charges we can omit q_i in the i-th equation, for all i. In this way we obtain a system of three homogeneous linear equations for three variables q_j. It is easy to explicitly write down the coefficients of this system. Since

$$F_{ji} = \frac{q_i q_j}{d_{ij}^3}(p_i - p_j)$$

where $p_i = \overrightarrow{OP_i}$ and $d_{ij} = \left|\left|p_i - p_j\right|\right|$, one easily verifies that this system has the form:

$$(F_{21}+F_{31}, T_1) = 0,\ (F_{12}+F_{32}, T_2) = 0,\ (F_{13}+F_{23}, T_3) = 0,$$

where $T_i=(-y_i, x_i)$. Writing down these equations after some obvious algebraic manipulations we get the following system of equations for the sought charges q_i:

$$\begin{pmatrix} 0 & (x_1y_2-x_2y_1)d_{31}^3 & (x_1y_3-x_3y_1)d_{21}^3 \\ (x_2y_1-x_1y_2)d_{32}^3 & 0 & (x_2y_3-x_3y_2)d_{12}^3 \\ (x_3y_1-x_1y_3)d_{23}^3 & (x_3y_2-x_2y_3)d_{13}^3 & 0 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}=0.$$

Notice that the expression $A_{ij}=x_iy_j-x_jy_i$ is (two times) the oriented area of the triangle ΔOP_iP_j. Obviously, $A_{ij}=-A_{ji}$. For this reason the matrix M of this system has a hidden skew-symmetric structure which yields a special factorization of M. Namely, by a direct computation one verifies that the matrix M of the above system can be factorized as $M = M'M''$, where M' is skew-symmetric. Since the determinant of any skew-symmetric matrix of odd order vanishes we conclude that $det\, M = 0$. So the system has non-trivial solutions which yield the sought stationary charges. In fact, this factorization can be written down explicitly as:

$$M'=\begin{pmatrix} 0 & x_1y_2-x_2y_1 & x_1y_3-x_3y_1 \\ x_2y_1-x_1y_2 & 0 & x_2y_3-x_3y_2 \\ x_3y_1-x_1y_3 & x_3y_2-x_2y_3 & 0 \end{pmatrix},\ M''=\begin{pmatrix} x_2y_3-x_3y_2+d_{23}^3 & 0 & 0 \\ x_3y_1-x_1y_3 & d_{31}^3 & 0 \\ x_1y_2-x_2y_1 & 0 & d_{12}^3 \end{pmatrix}.$$

Now it is also easy to verify that the rank of matrix M is two except the centrally symmetric triple and vertices of a regular triangle. So, generically stationary charges are defined up to a constant multiple while in the above two cases the space of solutions is two-dimensional. By examining the signs of coefficients of this system it is easy to show that the three stationary charges are positive if and only if the origin (centre of the circle) is inside each of the three triangles $\Delta P_1P_2P_3$ defined by the charges, which exactly means that the triple is balanced.

To establish stability of such a Coulomb triple one can use the general results on the Hessian matrix of a constrained critical point given in [9]. In our situation the necessary computations are routine but rather long. So, we omit them for the reason of space. They will be published elsewhere in a more general setting of arbitrary convex domain. Notice that the minimum of $E_{Q/P}$ in this case is neither non-degenerate, nor isolated because of rotational symmetry of the circle.

This result and its proof serve as a paradigm for our further considerations and have several interesting corollaries. First, as was noticed by the author of [5], the first part of the argument is applicable to n-tuples of points with arbitrary n. In more detail, given n-tuple of points in the unit circle this argument yields a system of n homogeneous linear equations for n values of stationary charges.

Let us consider first the case where n is odd. In this case, using the explicit formulae for the coefficients of this system it can be shown that, analogously to the case $n=3$, the matrix M of this system can be always factorized as $M = M'M''$, where M' is skew-symmetric. Since, for odd n the determinant of any skew-symmetric matrix vanishes, we conclude that $det\, M = 0$ and our system has non-trivial solutions which yield the stationary charges. For further reference we formulate the final conclusion explicitly.

Corollary 1. For odd n, any n-tuple of points in the unit circle has a non-trivial collection of n stationary charges.

It is easy to show that if stationary charges are all positive then the given n-tuple is balanced in the above sense. Analysis of the case n=5 gives good evidence that the converse is also true, which suggests a criterion which we formulate as a conjecture.

Conjecture 1. For odd n, n-tuple of points in the unit circle has positive stationary charges if and only if it is balanced.

Another plausible conjecture is concerned with the stability of Coulomb equilibria.

Conjecture 2. For odd n, any balanced n-tuple of points in the unit circle can be stabilized by a system of n positive charges.

For even $n=2k$, the situation is essentially different. Namely, it this case the determinant *det M* does not vanish identically; so, not all $2k$-tuples in the unit circle are Coulomb equilibria. Since the coefficients of matrix M are given by explicit formulas, the equation $det\ M = 0$ yields an analytic criterion of Coulomb equilibrium in terms of the coordinates of given points. This criterion is non-trivial already for $n=2$, since, as was mentioned, a pair of points on the unit circle is a Coulomb equilibrium if and only if this is a pair of antipodes. It is also easy to construct examples of $2k$-tuples with non-vanishing *det M* for any k. For $n=4$, matrix M has the form

$$\begin{pmatrix} 0 & (x_1y_2-x_2y_1)d_{41}^3d_{31}^3 & (x_1y_3-x_3y_1)d_{41}^3d_{31}^3 & (x_1y_1-x_4y_1)d_{41}^3d_{31}^3 \\ (x_2y_1-x_1y_2)d_{32}^3d_{42}^3 & 0 & (x_2y_3-x_3y_2)d_{12}^3d_{42}^3 & (x_2y_4-x_4y_2)d_{12}^3d_{34}^3 \\ (x_3y_1-x_1y_3)d_{23}^3d_{43}^3 & (x_2y_3-x_3y_2)d_{13}^3d_{43}^3 & 0 & (x_3y_4-x_4y_3)d_{13}^3d_{23}^3 \\ (x_4y_1-x_1y_4)d_{24}^3d_{34}^3 & (x_4y_2-x_2y_4)d_{14}^3d_{34}^3 & (x_4y_3-x_3y_4)d_{14}^3d_{24}^3 & 0 \end{pmatrix}.$$

Here again one can notice a certain hidden skew-symmetric structure of M due to the appearance of signed areas A_{ij} of triangles ΔOP_iP_j and construct a similar factorization $M = M'M''$, where M' is skew-symmetric. However, as is already shown in the case for $n=2$, *det M* does not vanish identically. From the above remarks it follows that, for arbitrary even n, vanishing of *det M* is equivalent to vanishing of a certain (not identically zero) differentiable function $R(P)$ in Cartesian coordinates of the given points. We formulate this conclusion as another corollary of Theorem 1.

Corollary 2. A $2k$-tuple of points P in the unit circle is a Coulomb equilibrium if and only if $R(P) = 0$.

The explicit expression for $R(P)$ is rather difficult to analyze already for 4-tuples; so, we do not have general results about Coulomb 2k-tuples. However, physical intuition and the results available for odd n suggest two plausible conjectures similar to the case of odd n.

Conjecture 3. A collection P of $2k$-points in the unit circle has positive stationary charges if and only if P is balanced and $R(P) = 0$.

Conjecture 4. Any balanced $2k$-tuple of points in the unit circle with $R(P) = 0$ is a stable Coulomb equilibrium for a system of positive charges.

Using explicit formulas for $R(P)$ one can construct and investigate certain classes of Coulomb quadruples. One can also describe the set of Coulomb quadruples as an explicit surface in three-dimensional Euclidean space and obtain further geometric information on the geometry of Coulomb quadruples. However, the necessary calculations are quite lengthy and involved. For $n=6$, it seems that further progress may be achieved using computer algebra packages.

Two natural directions of further research suggested by Theorem 1 have been indicated in [5]. The first one was to find out if analogs of Theorem 1 hold for more than three charges in the circle. The second one was to investigate its analogs for three charges on other convex curves, in particular, on an ellipse. Some comments on the first direction have been presented above. In the next section we briefly discuss the case of three charges on an ellipse different from the circle.

3. Let E be an ellipse parameterized by the arc-length $x(t) = a \cos t$, $y(t) = b \sin t$, $a > b > 0$. For a point $P(x,y)$ on E denote by $\tau = (-ay/b, bx/a)$ the unit tangent vector at P. To formulate the main result of this section it is convenient to use a notion introduced in [7]. Namely, let us say that a triple of points on E is an *orthotripod* if the normal lines to E at these points are concurrent (have a common point). We denote by $[E]$

the closure of interior domain of E and use it in the role of conductor X.

Theorem 2. A triple of points in $[E]$ is a Coulomb equilibrium if and only if it is either an orthotripod or all points lie on one of the axes of E. The critical charges are all positive if and only if the triangle $\Delta P_1P_2P_3$ contains the origin. In these case the corresponding equilibrium is stable.

The proof follows the same lines as for Theorem 1. Using the above formulae for the resultant forces and the unit tangent vector one writes the condition of equilibrium $(F_i, \tau_i) = 0$, i=1, 2, 3. This immediately yields an explicit form of the matrix of linear system for stationary charges. It is then possible to calculate its determinant and write down an analytic criterion for its vanishing. After that some algebraic manipulations enables one to show that the condition of its vanishing is equivalent to the algebraic criterion for the concurrence of the three normals in question. The details of the proof are purely computational and therefore omitted.

The fact that the marginal Coulomb equilibria are exactly the orthotripods has been proven in [7] in an essentially different geometric way. It should be added that the argument used in [7] is more general and applicable for any smooth convex curve. However the problem of stability has not been considered in [7] and it seems unlikely that it could be successfully treated using the geometric methods of [7].

The results presented above and in [5, 7] enable one to formulate and solve a number of concrete problems concerned with Coulomb equilibria on convex curves. These developments require a separate presentation so we do not describe them here and finish this note with a simple result on three charges in the unit ball.

4. Denote by B the unit ball in three-dimensional Euclidean space and by S its boundary, the unit sphere. The next result gives a description of possible equilibrium configurations of three point charges confined to the ball B.

Theorem 3. A triple of points in B is a Coulomb equilibrium if and only if they lie either on the same diameter or on the same great circle in S. If T is a balanced triple on a great circle then it has positive stationary charges and the corresponding equilibrium is stable.

The proof is straightforward. First, arguing as in the proof of Theorem 1 we show that a triple can be a Coulomb equilibrium in B if and only if it is condiametric or marginal. The condiametric case is obvious. For a marginal triple, one considers a similar system of linear equations for the stationary charges. If N_j denotes the unit normal to S at point P_j then the criterion of Coulomb equilibrium can be expressed by vanishing of vector-products $[F_i, N_i]$ at each point of the triple. This yields an explicit system of linear equations and the desired conclusion is obtained by analyzing the rank of this system. Details are again routine and we omit them.

Next natural step would be to characterize Coulomb triples in an ellipsoid X. Obvious candidates for Coulomb triples are triples lying on any of coordinate axis and orthotripods in any of the three "equatorial ellipses" (intersections of the surface of ellipsoid X with coordinate planes). Our results show that such triples are indeed Coulomb equilibria. We have good evidence that this is a complete description of Coulomb equilibria in an ellipsoid but we do not have a rigorous proof.

In conclusion, we wish to mention that there are many other natural perspectives of further research, in particular, concerned with Coulomb quadruples on ellipsoids and analogous problems in higher dimensions.

მათემატიკა

წერტილობრივი მუხტების წონასწორობები ამოზნექილ არეებში

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წერტილთა სამეულისათვის ნაჩვენებია წერტილობრივი მუხტების წონასწორული მდგომარეობის არსებობა და მიღებულია კრიტერიუმი და ანალიზური გამოსახულება ასეთი კონფიგურაციისათვის. ეს კრიტერიუმი გამოყენებულია წერტილობრივი მუხტების წონასწორული კონფიგურაციების აღსაწერად წრეწირსა და ელიფსზე. განხილულია აგრეთვე მომიჯნავე პრობლემები და ამოცანის შესაძლო განზოგადებები.

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