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Mathematics

Algebraic Function Fields and Non-Standard Analysis

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ABSTRACT. Non standard analysis was developed in 1960-1970 as an application of logical methods to such topic as abstract algebra, analysis field theory. The present work considers applications of non-standard analysis to modern algebra. It is based on A. Robinson's work – Algebraic Function Fields and Non-Standard Arithmetics. A possibility of generalizing Robinson's argument is considered. Some other applications of non-standard analysis to algebra include, for example, applications of non-standard methods, computational group theory and the theory of rewrite rules. *© 2015 Bull. Georg. Natl. Acad. Sci.*

Key words: *non-standard analysis, algebraic field theory, group theory, logic.*

Let Q^* be a non-standard model of the field of rational numbers Q. Let α be an element of Q^* which is not contained in Q. Then it is easy to verify that α is transedental over Q. Thus the field $A = Q(\alpha) \subset {}^{\bullet}Q^*$ is the field of rational functions with rational coefficients (i.e. algebraic function field).

An internal valuation of Q is given either by a non-standard prime number in Q^* or by a standard prime number in Q, or by the archimedean valuation of Q^*. We shall refer to these valuations as valuations of the first, second and third kind, respectively. Let A be an algebraic function field over Q which is also a subfield of Q^*. We shall see that in certain circumstances an internal valuation of Q^* induces a valuation of A and that all valuations of A can be obtained in this way. In order to avoid repetitions we shall prove the corresponding assertion immediately for the case where Q is replaced by any algebraic number field (finite algebraic extension of Q) K. In these cases K^* may for example be chosen as an ultrapower of K. An internal valuation of K now is given either by non-standard prime ideal in K^* or by canonical extension to K^* of a standard prime ideal in K, or by canonical extension to K^* of an archimedean valuation of K. We shall refer to these valuations again as valuations of the first, second and third kind, respectively.

Let K be an algebraic number field, K^* a non-standard model of K and of an algebraic function field over K which is contained in K^*. We consider valuations of the first kind in K^*. Thus, let p be a non-standard prime ideal in K^*. For any $x \in K^*$. We denote by $v_p x$ the order of x at p. Then $v_p x \in Z^*$. Suppose now that v_p is not identically zero on $A-\{0\}$. Choose $\alpha \in A$ such that $v_p \alpha > 0$ $(\alpha \neq 0)$. Let $\beta \neq 0$ be another element of A. Then we shall show that $\frac{v_p \beta}{v_p \alpha}$ is a standard rational number.

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We observe first that $\alpha \notin K$ since the non-standard ideal p cannot divide any standard number in K^* (i.e. $v_p x = 0$ on $K - \{0\}$). Accordingly, if β is algebraic over $K(\alpha)$ there exists a non-zero polynomial $f(x, y) \in K[x, y]$. Let

$$f(x, y) = \sum c_{ij} x^i y^j \in K \tag{1.1}$$

since $f(\alpha, \beta) = 0$, there exist distinct non-zero terms $c_{ij} x^i y^j$ and $c_{kl} x^k y^l$ such that

$$v_p\left(c_{ij}\, \alpha^i \beta^j\right) = v_p\left(c_{kl} \alpha^k \beta^k\right) \tag{1.2}$$

but c_{ij} and c_{kl} belong to K, so $v_p c_{ij} = v_p c_{kl} = 0$ and $(i-k) v_p \alpha = (l-j) v_p \beta$. Now $l - j \neq 0$ for if $l - j = 0$ then $l - j = 0$ and $i - k \neq 0$, contrary to our choice of $v_p \alpha = 0$. Hence,

$$v_p \beta = \frac{i-k}{l-j} v_p \alpha\ . \tag{1.3}$$

Thus, $v_p \beta$ is a standard rational multiple of $v_p \alpha$. Putting

$$\omega_p x = \frac{v_p x}{v_p \alpha} \tag{1.4}$$

it is clear that (1.4) defines a valuation of A in the additive group of rational numbers, with $w_p x = 0$ for $x \in K - \{0\}$. Thus, w_p is a valuation of the algebraic function field A over K. It follows that w_p is discrete. We shall say that w_p (or any equivalent valuation) is induced by n_p in A. The valuation is up to equivalence independent of our particular choice of a.

Next, consider valuation of the second kind in K^*. Let the p be a standard prime ideal in K^* (i.e. canonical extension of prime ideal of K to K^*). Using the same notation as before suppose now that v_p is not finite (in non-standard sense) for some $x \in A - \{0\}$. Choose $\alpha \in A$ such that $v_p \alpha$ is positive infinite $(\alpha \neq 0)$. Then $\alpha \notin K$ since $v_p x$ is finite on $K - \{0\}$. Let $\beta \neq 0$ be an element of A. As before there exists a polynomial $f(x, y) \in K[a, y]$ as given by such that $f(\alpha, \beta) = 0$. Hence again (1.2) is satisfied for two distinct non-zero terms of $f(x, y)$. From (1.2),

$$(i-k) v_p \alpha = (l-j) v_p \beta + \left(v_p c_{kl} - v_p c_{ij}\right). \tag{1.5}$$

Now $l - j = 0$ would imply that $v_p \alpha$ is finite, which is contrary to our choice of α. Hence, $l - j \neq 0$ and

$$\frac{v_p \beta}{v_p \alpha} = \frac{i-k}{l-j} + \frac{v_p c_{ij} - v_p c_{kl}}{(l-j) v_p \alpha}\ . \tag{1.6}$$

The numerator in the second term on the right-hand side of (1.6) is finite while the denominator is infinite. It follows that the term is infinitesimal. Hence, taking standard parts on both sides of (1.6). We obtain, introducing $\omega_p \beta$ by definition

$$\omega_p \beta = {}^{\circ}\!\left(\frac{v_p \beta}{v_p \alpha}\right) = \frac{i-k}{l-j}\,. \tag{1.7}$$

Notice that this formula also contains some additional information of the proximity of ${}^{\circ}\!\left(\frac{v_p \beta}{v_p \alpha}\right)$ to $\frac{i-k}{l-j}$ since it shows that the difference between these quantities cannot exceed some finite multiple of $(v_p \alpha)^{-1}$

(or, equivalently if $v_p\beta \neq 0$ some finite multiple of $\left(v_p\beta\right)^{-1}$).

We claim that $\omega_p x$ as given by (1.7) is a valuation of A over K in the additive group of standard rational numbers. Indeed, for $\beta \neq 0$, $\omega_p\beta = 0$ is standard rational, by definition. Also, for $\beta = K - \{0\}$, $\omega_p\beta = 0$ since $v_p\beta$ is then finite. Also, for $\beta, \gamma \in A$, $\beta \neq 0$, $\gamma \neq 0$,

$$\omega_p(\beta\gamma) = {}^{\circ}\left(\frac{v_p(\beta\gamma)}{v_p\alpha}\right) = {}^{\circ}\left(\frac{v_p\beta + v_p\gamma}{v_p\alpha}\right) = {}^{\circ}\left(\frac{v_p\beta}{v_p\alpha}\right) + {}^{\circ}\left(\frac{v_p\gamma}{v_p\alpha}\right) = \omega_p\beta + \omega_p\gamma,$$

which shows that the rule for the value of a product is satisfied. To consider the value of a sum, we may suppose that $v_p\beta \leq v_p\gamma$ and $\beta + \gamma \neq 0$, then we have $\omega_p\beta \leq \omega_p\gamma$ and

$$\omega_p(\beta + \gamma) = {}^{\circ}\left(\frac{v_p(\beta+\gamma)}{v_p\alpha}\right) \geq {}^{\circ}\left(\frac{v_p\beta}{v_p\,\alpha}\right) = \omega_p\beta.$$

Thus, ω_p is a valuation of A over K. We call it the valuation which is induced by v_p in A. The valuation is up to equivalence independent of our particular choice of α.

Finally, consider valuations of the third kind in K^*. Such a valuation is given by the absolute values of $|x|$ provided by an embedding of K^* in the corresponding non-standard model of the complex numbers C^*. The value of $|x|$ depends on the embedding but we shall not indicate this in the notation.

Suppose that $|x|$ does not remain finite everywhere on A. That is to say, $|x|$ is infinite for some $x \in A$ and hence, considering inverses that it is infinitesimal but not zero somewhere on A. We choose a fixed $\alpha \neq 0$ in A such that $|\alpha|$ is infinitesimal. Then $\alpha \notin K$. Let $\beta \neq 0$ be an element of A. Again there exists a polynomial $f(x,y) \in K[x,y]$ as given by (2.1) such that $f(\alpha,\beta) = 0$. Pick a term $c_{ij}x^j y^j$ such that $\left|c_{ij}\,\alpha^i\beta^j\right|$ is as large as possible. Then $\left|c_{ij}\,\alpha^i\beta^j\right| \neq 0$ and for any other term $c_{kl}x^k y^l$ the ratio

$$\frac{\left|c_{kl}\alpha^k\beta^l\right|}{\left|c_{ij}\alpha^i\beta^j\right|} \tag{1.8}$$

cannot exceed 1. At the same time, this ratio cannot be infinitesimal for all such terms; for if it was then $\dfrac{\left|f(\alpha,\beta) - c_{ij}\alpha^i\beta^j\right|}{\left|c_{ij}\alpha^i\beta^j\right|}$ also would be infinitesimal. And this is impossible, for this ratio is actually equal to 1 since $f(\alpha,\beta) = 0$. Accordingly, there exists a monomial, $c_{kl}x^k y^l$ distinct from $c_{ij}x^i y^j$ such that the ratio (1.8) is not infinitesimal. It follows that the natural logarithm of (1.8) is finite real number $-\mu$ where $\mu \geq 0$,

$$\ln\left|c_{kl}\right| - \ln\left|c_{ij}\right| + (k-i)\ln|\alpha| + (l-j)\ln|\beta| = -\mu.$$

Now $\ln\left|c_{kl}\right|$ and $\ln\left|c_{ij}\right|$ are finite since c_{kl} and c_{ij} are standard while $\ln|\alpha|$ is negative infinite, since $|\alpha|$ is infinitesimal

$$(k-i)\ln|\alpha| + (l-j)\ln|\beta| = \nu,$$

where ν is finite, and $l - j \neq 0$, otherwise $\ln|\alpha|$ also would have to be finite. Then

$$\frac{\ln|\beta|}{\ln|\alpha|} = \frac{i-k}{l-j} + \frac{\nu}{(l-j)\ln|\alpha|}, \tag{1.9}$$

where the last term on the right-hand side is infinitesimal. We put

$$\omega\beta = {}^{\circ}\left(\frac{\ln|\beta|}{\ln|\alpha|}\right) = \frac{i-k}{l-j} \tag{1.10}$$

so that $\omega\beta$ is a standard rational number for all $\beta \in A$, $\beta \neq 0$ and we assert that this defines a valuation of *A* over *K*. Here again (1.9) gives a measure for the difference between $\omega\beta$ and $\frac{\ln|\beta|}{\ln|\alpha|}$ by showing that this difference cannot exceed a finite multiple of $\left(\ln|\alpha|\right)^{-1}$ (or if $\ln|\beta| \neq 0$, $\left(\ln\beta\right)^{-1}$). Let $\beta, \gamma \in A$, $\beta \neq 0$, $\gamma \neq 0$. Then

$$\omega(\beta\gamma) = {}^{\circ}\left(\frac{\ln|\beta\gamma|}{\ln|\alpha|}\right) = {}^{\circ}\left(\frac{\ln|\beta|}{\ln|\alpha|} + \frac{\ln|\gamma|}{\ln|\alpha|}\right) = {}^{\circ}\left(\frac{\ln|\beta|}{\ln|\alpha|}\right) + {}^{\circ}\left(\frac{\ln|\gamma|}{\ln|\alpha|}\right) = \omega\beta + \omega\gamma .$$

Also, assuming $-\ln|\beta| \leq -\ln|\gamma|$ and $\beta + \gamma \neq 0$, we have $\omega\beta \leq \omega\gamma$ and

$$\omega(\beta+\gamma) = {}^{\circ}\left(\frac{\ln|\beta+\gamma|}{\ln|\alpha|}\right) = {}^{\circ}\left(\frac{\ln|\beta|}{\ln|\alpha|} + \frac{\ln\left|1+\frac{\gamma}{\beta}\right|}{\ln|\alpha|}\right). \tag{1.11}$$

Since $\beta + \gamma \neq 0$ and $1 + \frac{\gamma}{\beta} \in A$, $\frac{\ln\left|1+\frac{\gamma}{\beta}\right|}{\ln|\alpha|}$ is finite. We claim that

$${}^{\circ}\left(\frac{\ln\left|1+\frac{\gamma}{\beta}\right|}{\ln|\alpha|}\right) \geq 0 . \tag{1.12}$$

Since $\ln|\alpha|$ is negative infinite (1.12) will be established if we can show that $\ln\left|1+\frac{\gamma}{\beta}\right|$ cannot be positive infinite, i.e., that it is either finite or negative infinite. By assumption, $\ln|\beta| - \ln|\gamma| \geq 0$, i.e. $\left|\frac{\gamma}{\beta}\right| \leq 1$. This shows that $\ln\left|1+\frac{\gamma}{\beta}\right| \leq \ln 2$ and proves (1.12). Hence, from (1.11)

$$\omega(\beta+\gamma) \geq {}^{\circ}\left(\frac{\ln|\beta|}{\ln|\alpha|}\right) = \omega\beta .$$

Finally, since $\ln|\beta|$ is finite for all $\beta \in K - \{0\}$ $\omega\beta = 0$ for all such β. This shows that ωx is a valuation of *A* over *K*. We shall say that ωx is induced by the given archimedean valuation. Once again, ωx is up to equivalence independent of the particular choice of α.

Examples which show that all three kinds of induced valuations may actually occur can be obtained already for $K = Q$, $A = Q(\omega)$, $\omega \in Q^{*} - Q$. Choosing ω as a non-standard prime number in Q^{*} we see that $p = (\omega)$

yields a valuation of the first kind which induces a valuation of A choosing $\omega = p^{\nu}$ where p is a standard rational prime and ν is a non-standard positive integer, we obtain a valuation of the first kind which induce a valuation of A from $p = (\omega) = (p^{\nu})$. Finally, for $\omega = \nu^{-1}$, where ν is an arbitrary non-standard positive integer, the archimedean valuation of Q^* also induces a valuation of A. However, it is entirely possible that different valuations of K^* and even valuations of different kinds may induce the same valuation of A.

Now that K be an algebraic number field as before, suppose first that $A = K(\omega) \subset K^*$, where $\omega \in K^* - K$ so that $K(\omega)$ is the field of rational functions of one variable with coefficients in K. We require the following Theorem.

Theorem 1.1. *Let α be any algebraic integer which belongs to K^*. If $|\alpha|$ is finite in all archimedean valuations of K^* then α must be standard, $\alpha \in K$.*

Proof. Choose a fixed embedding of K^* in the corresponding non-standard model of the complex numbers C^*. Let $\alpha = \alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(n)}$ be the conjugates of α in C^*. Suppose that $\left|\alpha^{(1)}\right|, \left|\alpha^{(2)}\right|, \ldots, \left|\alpha^{(n)}\right|$ are finite. Then α has to be standard because let b be a finite upper bound for $\left|\alpha^{(i)}\right|$, $i = 1, \ldots, n$, and $s_1, \ldots, s_n$ be the fundamental symmetrical functions of $\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(n)}$. Then $|s_k| \le \binom{n}{k} b$, $k = 1, \ldots, n$, so the s_k are finite rational integers in Q^*. They are therefore standard, $s_k \in Q$, $k = 1, \ldots, n$, consider the polynomial

$$f(x) = x^n - s_1 x^{n-1} + \cdots + (-1)^n s_n .$$

This is a standard polynomial whose roots are therefore also standard. But α is one of these roots. Accordingly α is standard. This proves the assertion. □

Thus we can see, that a nonstandard valuations induce standard ones. The transcendence of nonstandard number over some field of standard numbers was crucial for this argument, but at the same time it was not derived from any "standard" situation involving transcendental numbers, which could be seen as an indication of usefulness of nonstandard methods.

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ალგებრული ფუნქციური ველები და არასტანდარტული ანალიზი

ა. კლიმიაშვილი

საქართველოს ტექნიკური უნივერსიტეტი

(წარმოდგენილია აკადემიის წევრის ხ. ინასარიძის მიერ)

ნაშრომი ეხება არასტანდარტული ანალიზის გამოყენებას თანამედროვე ალგებრაში. არასტანდარტული მეთოდები გამოიყენება ალგებრულ ველებსა და არასტანდარტულ

არითმეტიკასთან მიმართებაში. ეს მეთოდები ეფუძნება ა. რობინსონის მიერ ჩამოყალიბებულ არასტანდარტული ანალიზის თეორიას. რობინსონმა შეისწავლა ასევე არასტანდარტული ალგებრული ველები და მიიღო არასტანდარტულ ანალიზზე დაფუძნებული შედეგები ფუნქციური ალგებრული ველებისა და ჯგუფთა თეორიისთვის. მოცემული სტატია ეხება ფუნქციური ალგებრული ველებისთვის არასტანდარტული მეთოდების გამოყენებას.

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