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Mathematics

Approximation by Trigonometric Polynomials of Fractional Derivatives of Periodic Functions and the Properties of Conjugate Functions in $L^{p(x)}$ Spaces when min $p(x)$=1

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ABSTRACT. The present paper studies the problem of approximation of 2π -periodic functions and the properties of conjugate functions in the spaces $L^{p(x)}$, when min $p(x)$=1. The Bernstein-Zygmund type inequality for fractional derivatives of trigonometric polynomials is established and relying on this inequality, the direct and inverse inequalities for fractional derivatives are obtained. The condition ensuring belonging of conjugate functions to the space $L^{p(x)}$, min $p(x)$=1 is explored. The Zygmund type inequality for generalized moduli of smoothness of conjugate functions is presented and a subclass, invariant with respect to the conjugate operator, is determined. *© 2015 Bull. Georg. Natl. Acad. Sci.*

Key words: *variable exponent Lebesgue spaces, generalized moduli of smoothness, fractional derivatives, conjugate functions.*

In the present paper we focus our attention on the problems of approximation of 2π -periodic functions in variable exponent Lebesgue spaces $L^{p(x)}$ under the condition that $\min\limits_{[-\pi,\pi]} p(x)=1$. Moreover, in the above-mentioned spaces we present the Zygmund type inequality for generalized moduli of smoothness of conjugate functions and on the basis of that inequality we find a subclass of the space $L^{p(x)}$, $\min\limits_{[-\pi,\pi]} p(x)=1$, which is invariant with respect to the conjugate operator.

The paper is organized as follows. Section 1 involves definitions and results which will be needed in the sequel. The Bernstein type inequality for fractional derivatives of trigonometric polynomials is given in Section 2. Next, the direct and inverse (in S. Bernstein's terms) inequalities of constructive theory of functions for fractional derivatives of 2π -periodic functions are established. In Section 3 we explore a condition which leads to the belonging of the conjugate to the function $f \in L^{p(x)}$, $\min\limits_{[-\pi,\pi]} p(x)=1$ to the same space; the Zygmund type inequality for generalized moduli of smoothness of conjugate functions is proved and a subclass, invariant with respect to the operation of conjugation, is found.

1. Preliminaries.

Let $p(x)$ be a 2π-periodic function, continuous on the whole axis. Assume $T=[-\pi,\pi]$. By P_1 we denote a set of those $p(x)$ for which

$$1 \le \underline{p} \le p(x) \le \overline{p} < +\infty, \tag{1.1}$$

where $\underline{p} = \min_T p(x)$, $\overline{p} = \max_T p(x)$.

The set of exponents $p(x)$ for which $1 < \underline{p} \le p(x) \le \overline{p} < +\infty$ we denote by P. By $P^{\log}$ we denote a set of those $p(x)$ for which the log-continuity condition

$$|p(x)-p(y)| \le \frac{A}{-\ln|x-y|}, |x-y| < \frac{1}{2} \tag{1.2}$$

is fulfilled. Here the constant A does not depend on x and y from the set T.

The variable exponent Lebesgue $L^{p(x)}(T)$ spaces for the exponent $p \in P_1$ is the set of all measurable functions $f(x)$ for which

$$\|f\|_{p(\cdot)} := \inf\left\{\lambda > 0, \int_T \left|\frac{f(x)}{\lambda}\right|^{p(x)} dx \le 1\right\}.$$

It is known that $L^{p(x)}$ spaces are the Banach function spaces (see, for example, [1]).

Throughout the paper, we will employ the following structural and constructive characteristics of 2π-periodic functions from $L^{p(x)}(T)$:

$$\Omega(f,\delta)_{p(\cdot)} = \sup_{0<h\le\delta} \left\| \frac{1}{2h}\int_{x-h}^{x+h} f(t)\,dt - f(x) \right\|_{p(\cdot)} \tag{1.3}$$

is a generalized moduli of smoothness, and the best approximation by trigonometric polynomials

$$E_n(f)_{p(\cdot)} = \inf \|f - \mathrm{T}_k\|_{p(\cdot)}, \tag{1.4}$$

where the infinitum is taken under all trigonometric polynomials of degree not greater than n.

In [2-3], under the condition $p \in P^{\log} \cap P$, the authors prove the refined direct and inverse theorems of constructive theory of functions, including weighted variable exponent Lebesgue spaces and periodic differentiable in a generalized sense functions. Here we mean differentiability in a sense introduced in [4].

In the case where $p \in P^{\log} \cap P_1$, i. e. $\min_{[-\pi,\pi]} p(x) = 1$, the direct theorem (an analogue of the second direct Jackson's theorem) for derivative of integral order, has been indicated in [5-6]. However, in the latter paper there is presented the theorem of inverse type for the class of Lipschitz type. In this work we, firstly, extend the result obtained in [5] to fractional derivatives and, secondly, we have obtained a general inverse theorem for fractional derivatives.

Another problem in the paper under consideration concerns both the condition ensuring belonging to the space $L^{p(x)}$, $\min_{[-\pi,\pi]} p(x) = 1$, of a conjugate to the function $f \in L^1$ and the classes, invariant with respect to the conjugate operator.

In [7], S.B. Stechkin in terms of the best approximations in the space C has found the condition for the conjugate function to be continuous. In the same work it is mentioned that analogous results are likewise valid for the classical Lebesgue space L^1. In the present work we solved the same problem in a general statement, in the framework of spaces $L^{p(x)}$, $p \in P^{\log} \cap P_1$.

2. Direct and Inverse Theorems for α -Order Derivatives of Periodic Functions

In this section we will employ the derivative of fractional α -order α >0 in Weyl's sense whose definition is given, e. g., in A. Zygmund's book (Vol. II, Ch. XII, Sect. 8). For the 2π -periodic function $f \in L^1(T)$, the above-mentioned derivative we denote by $f^{(\alpha)}$.

The basis of our investigation in this section is the following Bernstein-Zygmund type theorem for fractional order derivatives of trigonometric polynomials.

Theorem 1.1. *Let* $p \in P^{\log} \cap P_1$ *and* α >0. *Then for trigonometric polynomials* T_n *the inequality*

$$\left\| \mathrm{T}_n^{(\alpha)} \right\|_{p(\cdot)} \le cn^{\alpha} \left\| \mathrm{T}_n \right\|_{p(\cdot)} \tag{2.1}$$

holds, where the constant c does not depend on T_n and n.

Note that the inequality of type (2.1) for $p \in P^{\log} \cap P$ and for general type derivatives in weighted setting has been proved in [2]. Certain improvements of the corresponding inequaliy is given in [8]. In [5] and [6], the inequality of type (2.1) is stated only for positive integer α , but under the condition $\underline{p} = 1$.

By $W^{p(\cdot),\alpha}$ we denote a set of those 2π -periodic functions for which

$$\|f\|_{W^{p(\cdot),\alpha}} = \|f\|_{p(\cdot)} + \left\| f^{(\alpha)} \right\|_{p(\cdot)} < +\infty .$$

The following statements are valid:

Theorem 2.2. *Let* $p \in P^{\log} \cap P_1$ *and if* $f \in W^{p(\cdot),\alpha}$, $\alpha \ge 0$, *then the inequality*

$$E_n(f)_{p(\cdot)} \le \frac{c}{n^{\alpha}} \Omega\left(f^{(\alpha)}, \frac{1}{n} \right)_{p(\cdot)}, \tag{2.2}$$

holds, where the constant c does not depend on T_n and n.

Theorem 2.3. *Let* $p \in P^{\log} \cap P_1$, *and for some* α >0 *the condition*

$$\sum_{n=1}^{\infty} n^{\alpha-1} E_n(f)_{p(\cdot)} < +\infty$$

is fulfilled. Then $f \in W^{p(\cdot),\alpha}$, and the inequality

$$\Omega\left(f^{(\alpha)}, \frac{1}{n} \right)_{p(\cdot)} \le c\left(\frac{1}{n^2} \sum_{k=0}^{n} (k+1)^{\alpha+1} E_k(f)_{p(\cdot)} + \sum_{k=n+1}^{\infty} k^{\alpha-1} E_k(f)_{p(\cdot)} \right) \tag{2.3}$$

is valid.

Analogous statements are valid for generalized moduli of higher order smoothness.

3. The Condition of Belonging of a Conjugate Function to the Class $L^{p(\cdot)}$, $p \in P^{\log} \cap P_1$, Zygmund Type Inequality and Invariant Classes.

At the beginning of this section, in terms of the best approximations we indicate the condition for the function $f \in L^{p(\cdot)}$, $p \in P^{\log} \cap P_1$ which guarantees that the derivative of the conjugate function $\tilde{f}$ belongs to the same space.

Theorem 3.1. *Let* $p \in P^{\log} \cap P_1$ *and* $Z_0 = \{0,1,2,\ldots\}$. *If the condition*

$$\sum_{n=1}^{\infty} n^{r-1} E_n(f)_{p(\cdot)} < +\infty, \qquad r \in Z_0$$

is fulfilled, then the r-th order derivative of the conjugate function $\tilde{f}^{(r)} \in L^{p(\cdot)}$ and the inequality

$$E_n\left(\tilde{f}^{(r)}\right) \le c\left((n+1)^r E_n(f)_{p(\cdot)} + \sum_{k=n+1}^{\infty} k^{r-1} E_k(f)_{p(\cdot)}\right) \tag{2.4}$$

holds, where the constant c does not depend on f and n.

From the above theorem, in particular for $r = 0$, using the inverse theorem, we can conclude that the following assertion is valid.

Theorem 3.2. *Let* $p \in P^{\log} \cap P_1$. *If the condition*

$$\int_0^{\pi} \frac{\Omega(f,t)_{p(\cdot)}}{t} dt < \infty \tag{2.5}$$

is fulfilled, then $\tilde{f} \in L^{p(\cdot)}$, and Zygmund type inequality

$$\Omega\left(\tilde{f},\delta\right)_{p(\cdot)} \le c\left(\int_0^{\delta} \frac{\Omega(f,t)_{p(\cdot)}}{t} dt + \delta^2 \int_{\delta}^{\pi} \frac{\Omega(f,t)_{p(\cdot)}}{t^3} dt\right) \tag{2.6}$$

holds.

From this theorem we have the following

Corollary 3.1. *If for some* $k \in N$,

$$\int_0^{\pi} \frac{\Omega(f,t)_{p(\cdot)}}{t}\left(\ln\frac{2\pi}{t}\right)^k dt < \infty, \tag{2.7}$$

then

$$\int_0^{\pi} \frac{\Omega(\tilde{f},t)_{p(\cdot)}}{t}\left(\ln\frac{2\pi}{t}\right)^{k-1} dt < \infty.$$

By V_k we denote the set of those $f \in L^{p(\cdot)}$, $p \in P^{\log} \cap P_1$ for which condition (2.7) is fulfilled. Then Corollary 3.1 implies that if $f \in V_k$, then $\tilde{f} \in V_{k-1}$.

Assume

$$V = \bigcap_{k=0}^{\infty} V_k .$$

Then the validity of the following assertion is obvious.

Theorem 3.3. *Let* $p \in P^{\log} \cap P_1$. *Then the class* V *is invariant with respect to the conjugate operator.*

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მათემატიკა

პერიოდულ ფუნქციათა წილადური რიგის წარმოებულების აპროქსიმაცია ტრიგონომეტრიული პოლინომებით და ზიგმუნდის ტიპის უტოლობა $L^{p(x)}$ სივრცეში, როცა $\min p(x)=1$

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გამოკვლეულია 2π -პერიოდული ფუნქციის ტრიგონომეტრიული პოლინომებით აპროქსიმაციისა და შეუღლებული ფუნქციის თვისებები ცვლადმაჩვენებლიან $L^{p(x)}$ ლებეგის სივრცეებში, როცა $p(x)$ აკმაყოფილებს ლოგარითმული უწყვეტობის პირობას და $\min p(x)=1$. დადგენილია ბერნშტეინ-ზიგმუნდის ტიპის უტოლობა ტრიგონომეტრიული პოლინომების წილადური რიგის წარმოებულებისათვის. ამის საფუძველზე მიღებულია პირდაპირი და შებრუნებული უტოლობები ფუნქციის წილადური რიგის წარმოებულებისათვის. ამასთანავე, მოძებნილია პირობები, რომლებიც განაპირობებენ შეუღლებული ფუნქციისა და მისი წარმოებულების მიკუთვნებას $L^{p(x)}$ სივრცისადმი. შეუღლებული ფუნქციის განზოგადებული სიგლუვის მოდულისათვის დადგენილია ზიგმუნდის ტიპის უტოლობა და ამ უტოლობაზე დაყრდნობით აგებულია $L^{p(x)}$ სივრცის ინვარიანტული ქვესიმრავლე შეუღლების ოპერატორის მიმართ.

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