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Mathematics

On Locally Cyclic and Distributive Modules and Algebras

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(Presented by Academy Member Hvedri Inassaridze)

ABSTRACT. In the theory of abelian groups the following two results are fundamental: on the representation of finitely generated abelian groups in the form of a direct sum of cyclic subgroups and the classification of locally cyclic groups. The generalization of the first of them for the case of modules over the principal ideal domains is classical. However, there are no published works for locally cyclic modules over the principal ideal domains. The aim of this paper is to fill this gap, namely, classification locally cyclic modules over the principal ideal domains. *© 2015 Bull. Georg. Natl. Acad. Sci.*

Key words: *distributive lattice, locally cyclic module, principle ideal domain.*

Necessary Definitions

Let L be a lattice and $0, I \in L$. The lattice L is called distributive, if for any $x, y, z \in L$, one of the equations holds:

(i) $x \cup (y \cap z) = (x \cup y) \cap (x \cup z)$;

(ii) $x \cap (y \cup z) = (x \cap y) \cup (x \cap z)$.

The lattice with five elements (see Fig. 1) is called Diamant. The Lattice L is distributive if and only if it does not contain diamant [1].

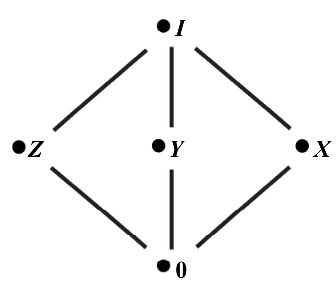

Fig. 1. Diamant.

Now, recall some necessary definitions and facts. Lattice L is torsion free $0, I \in L$, if none of the elements of L covers 0. Let A (not necessarily associative) be an algebra over the ring $\mathbb{R}$. Algebra A is torsion free if $\alpha a = 0$, $\alpha \in \mathbb{R}$, $a \in A$, implies that either $\alpha = 0$ or $a = 0$. Algebra A is called proper if for any $x \in A$, $x \neq 0$, lattice $L(\langle x \rangle)$ is torsion free. It is clear that proper algebra is torsion free. Torsion free algebra is not always proper. If $\mathbb{R}$ is a field and A is a Lie algebra over $\mathbb{R}$, then A is torsion free but not proper. For any $x \in A$, the lattice $L(\langle x \rangle)$ is with only two elements 0 and I. The $\mathbb{R}$ module X is periodical, if for any $x \in X$ $\operatorname{Ann}(x) = \operatorname{id}(\alpha)$, $\alpha \neq 0$, $\alpha \in \mathbb{R}$.

One-generated subalgebra is called cyclic. An algebra is called locally cyclic, if any finite set of its elements generate a cyclic subalgebra.

Distributive Lattice of Divisors

Let $\mathbb{R}$ be a commutative principal ideal domain. We consider the case when $\mathbb{R}$ is not a field. The case when $\mathbb{R}$ is a field is trivial. Let $P = \{P_\mu, \mu \in \Lambda\}$ be the set of all prime elements of $\mathbb{R}$. Let $E = \{\varepsilon_\alpha, \alpha \in \Lambda\}$ be the set of all units (invertible elements) of $\mathbb{R}$. Denote by $\mathfrak{A}$ the equivalent classes $\mathsf{U} = [\mathsf{U}_\mu, \mu \in \Omega]$ as follows:

$$P_{\mu_1}, P_{\mu_2} \in \mathsf{U}_\mu \Leftrightarrow P_{\mu_1} = \varepsilon P_{\mu_2}, \quad \mu_1, \mu_2 \in \Omega, \ \varepsilon \in E.$$

Fix for each class U_μ, $\mu \in \Omega$, one prime element P_μ, $\mu \in \Omega$. Consider the set $\tilde{\mathbb{R}}_+$, which consists of all finite products of P_μ, $\mu \in \Omega$, i.e.,

$$\tilde{\mathbb{R}}_+ = \left\{P_{\alpha_1}^{k_1} P_{\alpha_2}^{k_2} \cdots P_{\alpha_n}^{k_n}, \ P_{\alpha_j} \in \mathsf{U}_{\mu_i}, \quad k_i \in \mathbb{N}\right\}.$$

Let $\mathbb{R}_+ = \{\tilde{\mathbb{R}}_+, 0, 1\}$. Define the partial order $\leq$ on $\mathbb{R}_+$ as follows:

$$\sup\{a,b\} = a \cup b \ \text{ is the biggest common divisor of } a,b \in \tilde{\mathbb{R}}_+,$$

$$\inf\{a,b\} = a \cup b \ \text{ is the smallest common multiple of } a,b \in \tilde{\mathbb{R}}_+.$$

Consequently, $\langle \mathbb{R}_+, \leq \rangle$ can be interpreted as a lattice with the last element 0 and the biggest element 1. Define this lattice by L_∞. For $\alpha \in \mathbb{R}_+$, we introduce the notation L_α for the interval $[\alpha, 1] \subset L_\infty$.

It is easy to see that L_∞ and L_α are distributive lattices and L_∞ is a lattice of all divisors α.

Let $\langle x \rangle$ be cyclic $\mathbb{R}$-module with conditions $\operatorname{Ann}(x) = \infty$, i.e. $\alpha\, x \neq 0$ for all $\alpha \neq 0$. Then it is obvious that for any $\beta \in \mathbb{R}$, there exists the submodule $\langle \beta x \rangle$ in $\langle x \rangle$ such that

$$\operatorname{Ann}\left(\frac{\langle x \rangle}{\langle \beta x \rangle}\right) = \operatorname{id}(\beta) \subset \mathbb{R}.$$

Conversely, each nontrivial $\mathbb{R}$-submodule of $\langle x \rangle$ has the same property. Consequently, if we define the map

$$\sigma : L_\infty \to L(\langle x \rangle), \quad \sigma(l_\alpha) = \langle l_\alpha x \rangle, \quad l_\alpha \in \mathbb{R}.$$

We can see that σ is bijection and for all $l_s, l_t \in L_\infty$, it holds:

$$l_s \leq l_s \Leftrightarrow l_s \ \text{ is divisor of } l_t$$

Consequently, σ is lattice isomorphism.

If now $\operatorname{Ann}(x) = \operatorname{id}(\beta) \subset \mathbb{R}$, $\beta \neq 1$, then for any divisor of β (suppose μ) in the $\mathbb{R}$-module $\langle x \rangle$ there exists $\mathbb{R}$-submodule $\langle \mu\, x \rangle$ with the condition:

$$\operatorname{Ann}\left(\frac{\langle x \rangle}{\langle \mu x \rangle}\right) = \operatorname{id}(\mu) \subset \mathbb{R}, \quad \mu \neq 1.$$

Similarly, we can construct the bijection:

$$\sigma : L_\mu \to L(\langle x \rangle), \ \sigma(l_\mu) = \langle l_\mu x \rangle,$$

with the similar condition

$$l_{\mu_1} \leq l_{\mu_2} \Leftrightarrow \ \text{divides } \mathrm{I}_{\mu_2}.$$

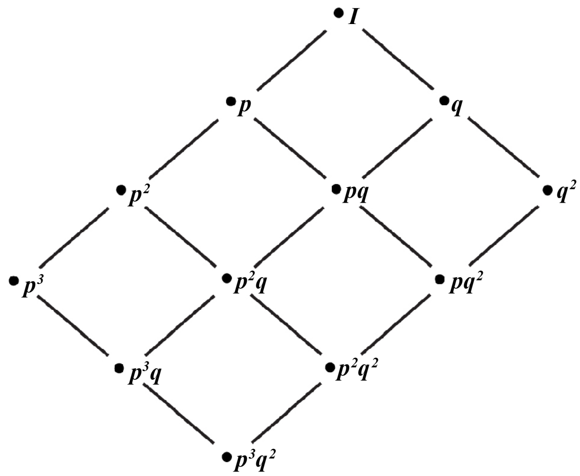

Fig. 2. Lattice Lp^3q^2.

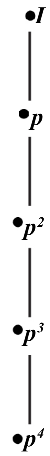

Fig. 3. Lattice Lp^4.

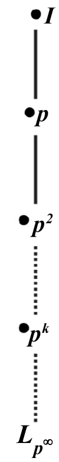

Fig. 4. Lattice L_p^∞.

This holds if and only if

$$\left\langle l_{\mu_1}, x\right\rangle \subset \left\langle l_{\mu_2}, x\right\rangle.$$

So we conclude that σ is isomorphism and it holds:

Theorem 1. *If* $\alpha \in \mathbb{R} \cup \infty$, $\operatorname{Ann}(\langle x\rangle) = \operatorname{id}(\alpha)$, *then*

(i) $L(\langle x\rangle) \cong L_\alpha$ if $\operatorname{Ann}(x) = \operatorname{id}(\alpha) \subset \mathbb{R}$;

(ii) $L(\langle x\rangle) \cong L_\infty$ if $\operatorname{Ann}(x) = \infty$.

Remark 1. For the torsion free element $x \neq 0$, we suppose that $\operatorname{Ann}(x) = \infty$.

Locally Cyclic Modules

Let $\mathcal{L}$ be a periodical Lie algebra over the principal ideal domain $\mathbb{R}$, i.e. $\forall\, x \in \mathcal{L}$, $\operatorname{Ann}(x) = \operatorname{id}(\alpha)$, $0 \neq \alpha \in \mathbb{R}$. Suppose $p_i \in \mathbb{R}$ is a prime element and $\mathcal{L}_{p_i}$ the set of all $x \in \mathcal{L}$, $\operatorname{Ann}(x) = \operatorname{id}\left(p_i^k\right)$, $k \in \mathbb{N}$. It is easy to see that $\mathcal{L}_{p_i}$ is an ideal in $\mathcal{L}$ and $\mathcal{L}$ has the decomposition in direct sum, i.e.

$$\mathcal{L} = \mathcal{L}_{p_1} \oplus \mathcal{L}_{p_2}\mathcal{L} \oplus \cdots \oplus \mathcal{L}_{p_\mu} \oplus \cdots. \qquad (*)$$

For the principal ideal domain $\mathbb{R}$ consider the quotient field, i.e.

$$\mathbb{R}_+ = \mathbb{R} \cup \mathbb{R}^{-1} = \left\{\frac{a}{b},\ a, b \in \mathbb{R},\ b \neq 0\right\}.$$

Theorem 2. $\mathbb{R}$*-module* $\mathbb{R}_+$ *is locally cyclic.*

Proof. Let us consider the finite set of elements $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_k}{b_k} \in \mathbb{R}_+$ and $\mathbb{R}$-submodule $\mathbb{M}$ generated by these elements. Submodule $\mathbb{M}$ consists of the elements

$$m_1 \cdot \frac{a_1}{b_1} + m_2 \cdot \frac{a_2}{b_2} + \cdots + m_k \cdot \frac{a_k}{b_k},$$

where m_i, $i = 1, \ldots, k$, are arbitrary elements of $\mathbb{R}$. These elements could be written in the form:

$$\frac{m_1 a_1 b_2 \cdots b_k + \cdots + m_k a_k b_1 \cdots b_{k-1}}{b_1 b_2 \cdots b_{k-1}}.$$

Consequently, all the elements of $\mathbb{R}_+$ have the form:

$$s \cdot \frac{1}{b_1 b_2 \cdots b_{k-1}} \text{ for some } s \in \mathbb{R},$$

i.e., they are in the cyclic $\mathbb{R}$-submodule $\left\langle \left(b_1 b_2 \cdots b_{k-1}\right)^{-1} \right\rangle$. So we can conclude that $\mathbb{R}$-module $\mathbb{R}_+$ is locally cyclic.

Remark 2. It is easy to see that the theorem is true for any commutative domain and its quotient field.

In $\mathbb{R}$-module $\mathbb{R}_+$, the elements of $\mathbb{R}$ can be considered as $\mathbb{R}$-submodule and, consequently, we can consider the factor-module $\mathbb{R}_+/\mathbb{R} = \mathbb{R}_+ (\operatorname{mod} \mathbb{R})$. This is periodic $\mathbb{R}$-module, i.e., for each:

$$r + \mathbb{R} \in \mathbb{R}_+ (\operatorname{mod} \mathbb{R}), \ \operatorname{Ann}(r + \mathbb{R}) = \operatorname{id}(\alpha), \ 0 \neq \alpha \in \mathbb{R}.$$

Really, if $\frac{a}{b} \in \mathbb{R}_+ (\operatorname{mod} \mathbb{R})$, i.e. $\frac{a}{b} = r + \mathbb{R}$. Then $b\left(\frac{a}{b}\right) \equiv 0 (\operatorname{mod} \mathbb{R})$, Consequently, as periodic $\mathbb{R}$-module, $\mathbb{R}_+/\mathbb{R}$ has representation in the form $(*)$. Each component $\mathcal{L}_{p_\mu}$ from $(*)$ contains the elements:

$$\frac{1}{p_i (\operatorname{mod} \mathbb{R})}, \frac{1}{p_i^2 (\operatorname{mod} \mathbb{R})}, \ldots, \frac{1}{p_i^k (\operatorname{mod} \mathbb{R})}, \ldots.$$

Each element of $\mathcal{L}_{p_i}$ has the form $\frac{m}{p^k}$, $(m, p) = 1$. Consequently, they are in this cyclic $\mathbb{R}$-module $\left\langle \frac{1}{p_i^k} \right\rangle$. So, any $\mathbb{R}$-submodule of $\mathcal{L}_{p_i}$ is either cyclic, or contains infinite elements of the set:

$$\frac{1}{p_i}, \frac{1}{p_i^2}, \ldots, \frac{1}{p_i^k}, \ldots.$$

Thus we can conclude that each submodule of $\mathcal{L}_{p_i}$ is either cyclic or coincides with $\mathcal{L}_{p_i}$. Consequently, if we take finite set of the elements from $(*)$, then it generates either cyclic $\mathbb{R}$-module, or finite direct sum of cyclic $\mathbb{R}$-modules from different components of $\mathcal{L}_{p_i}$. Thus we conclude that $\mathbb{R}$-module $\mathbb{R}_+ (\operatorname{mod} \mathbb{R})$ is again cyclic. So we proved:

Theorem 3. *Periodic $\mathbb{R}$-module $\mathbb{R}_+ (\operatorname{mod} \mathbb{R})$ is locally cyclic.*

Distributive Lattices of Submodules and Subalgebras

Now we shall prove the analogy of the theorem of O. Ore, which is one of the fundamental theorems in the lattice theoretical characteristic of groups.

Proposition 1. *Let Lie algebra $\mathcal{L}$ be defined over arbitrary commutative ring $\mathbb{R}$. If L is locally cyclic, then the lattice $L(\mathcal{L})$ is distributive.*

Proof. Let A, B, C be sub-algebras of L. We have $A \cap B \subseteq A$, $A \cap B \subseteq B \subseteq B \cup C$, thus $A \cap B \subseteq A \cap (B \cup C) = U$. We have as well $A \cap C \subseteq A$, $A \cap C \subseteq C \subseteq C \cup B$. Consequently,

$$V = (A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C) = U.$$

Let us prove the inverse inclusion. Consider the element:

$$g \in U, \ g = a = b + c, \ a \in A, \ b \in B, \ c \in C.$$

L is locally cyclic so $\langle b, c \rangle = \langle u \rangle$ and we have $qu = b$, $su = c$, $q, s \in \mathbb{R}$.

On the other hand,

$$mb + nc = u, \quad (mq + ns)u = u, \quad a = b + c = (q + s)u.$$

Let

$$\begin{aligned} x &= qa = q(q+s)u = (q+s)b \in A \cap B, \\ &= sa = s(q+s)u = (q+s)c \in A \cap C. \end{aligned}$$

Then

$$a = (q+s)u = (q+s)(me+ns)u = \left[mq(q+s)+ns(q+s)\right] = u = mx+ny.$$

Thus, we have $a \in (A \cap B) \cup (A \cap C)$.

On Application to Subalgebra Lattices

Now for principal ideal domains, we shall prove the theorem of O. Ore [4] (see also [5]).

Theorem 4. *Lie algebra $\mathcal{L}$ over the principal ideal domain $\mathbb{R}$ has distributive subalgebra lattice, if and only if it is locally cyclic.*

Proof. We will suppose $\mathbb{R}$ is not a field. For the case of field, the situation is trivial, i.e. if $L = \langle x \rangle \cup \langle y \rangle$, then $L(\mathcal{L})$ contains diamant (Fig. 1) $\langle x \rangle = X$, $\langle y \rangle = Y$, $\langle x+y \rangle = Z$.

Suppose there is finitely generated but not one-generated subalgebra A in L. As $L(\mathcal{L})$ is distributive, more than that it is modular. So if the lattice $L(\mathcal{L})$ is modular and $\mathbb{R}$ is principal ideal domain, which is not a field, then each submodule of L is subalgebra [2]. Consequently, L as a module over the principal ideal domain $\mathbb{R}$ is finitely generated. As $\mathbb{R}$-module, algebra A has the decomposition in direct sum [3], i.e.

$$A = \langle a_1 \rangle \oplus \langle a_2 \rangle \oplus \cdots \oplus \langle a_n \rangle,$$

where all $\langle a_i \rangle$, $i = 1, 2, \ldots, m$, are primary or free cyclic subalgebras. As A is not one-generated, among the $\langle a_i \rangle$, we can choose two $\langle a_j \rangle$, $\langle a_k \rangle$ with coprime periods. Now, if $\langle a_j \rangle = X$, $\langle a_k \rangle = Y$, $\langle a_j + a_k \rangle = Z$, then in the lattice $L(A)$ there exists diamant (Fig. 1). Consequently, we have a contradiction, i.e. $L(\mathcal{L})$ is not distributive. The inverse is a corollary of the Proposition 1.

Remark 3. The theorem does not give guarantee that from the distributivity of $L(\mathcal{L})$ to conclude that L is locally cyclic. There exist rings and non-locally cyclic modules over such rings with distributive submodule lattices.

Corollary 1. *Algebra Lie L over the principal ideal domain $\mathbb{R}$ is cyclic, if and only if, when the subalgebra lattice $L(\mathcal{L})$ is distributive lattice with maximal condition.*

Proof. It is well-known that for Lie algebra L over the principal ideal domain $\mathbb{R}$ the subalgebra lattice $L(\mathcal{L})$ has maximal condition, then it is finitely generated. As the lattice $L(\mathcal{L})$ is distributive with maximal condition, then $\mathcal{L}$ is locally cyclic and finitely generated, thus it is cyclic. On the other hand, if L is cyclic, then $L(\mathcal{L})$ is distributive (Proposition 1). More than that, if $0 < H \le L$, then the lattice $L(\mathcal{L}/H)$ has finite length. It is clear that $L(\mathcal{L})$ satisfies maximal condition.

Theorem 5. *If L and L_1 are Lie algebras over the principal ideal domains $\mathbb{R}$ and $\mathbb{R}_+$, and $L(\mathcal{L}) \cong L(\mathcal{L}_1)$, then torsion free cyclic subalgebras are mapped to torsion free cyclic subalgebras and periodical cyclic subalgebras are mapped to periodical cyclic subalgebras.*

Corollary 2. *Algebra Lie L over the principal ideal domain $\mathbb{R}$, if and only if is proper cyclic when the lattice $L(\mathcal{L})$ is torsion free with maximal condition.*

Remark 4. To be torsion free lattice $L(\mathcal{L})$ is guaranteed by the fact that L has no torsion as a module and $\mathbb{R}$ is a principal ideal domain, which is not a field, i.e. $\mathcal{L}$ is a proper Lie algebra.

Remark 5. For Lie algebras over principal ideal domain being cyclic subalgebra is preserved under the lattice isomorphisms.

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მათემატიკა

ლოკალურად ციკლური და დისტრიბუციული მოდულები და ალგებრები

ა. ლაშხი

საქართველოს ტექნიკური უნივერსიტეტი

(წარმოდგენილია აკადემიის წევრის ხ. ინასარიძის მიერ)

აბელურ ჯგუფთა თეორიაში ორი შედეგია ფუნდამენტური: ერთი — სასრულ წარმომქმნელიანი აბელური ჯგუფების ციკლური ქვეჯგუფების პირდაპირ ჯამებად დაშლის შესახებ, ხოლო მეორე — ლოკალური ციკლური ჯგუფების კლასიფიკაცია. კლასიკურია პირველი მათგანის განზოგადება მთავარ იდეალთა რგოლებზე განსაზღვრული მოდულებისთვის. ლოკალურად ციკლური მოდულები მთავარ იდეალთა რგოლებზე კი კლასიფიცირებული არ არის. ნაშრომის მიზანია ამ ხარვეზის აღმოფხვრა, კერძოდ, მოცემულია ლოკალურად ციკლური მოდულების კლასიფიკაცია მთავარ იდეალთა რგოლებზე განსაზღვრული მოდულებისათვის.

REFERENCES

1. *Birkhoff G.* (1961) Lattice theory. Revised ed. American Mathematical Society Colloquium Publications.
2. *Lashkhi A.A.* (1985) Itogi Nauki i Tekhniki. Ser. Sovrem. Problem. Mat. Nov. Dostizh. **26**: 213-257.
3. *Lashkhi A.A.* (1975) Trudy Mat. Inst. AN GSSR **46**: 5-21 (in Russian).
4. *Lang S.* (2002) Algebra. Revised third edition. Graduate Texts in Mathematics. 211. Springer-Verlag, New York.
5. *Schmidt R.* (1994) Subgroup lattices of groups. De Gruyter Expositions in Mathematics. 14. Walter de Gruyter & Co., Berlin.
6. *Ore O.* (1937) I. Duke Math. J. **3**, 2: 149-174.
7. *Ore O.* (1938) II. Duke Math. J. **4**, 2: 247-269.
8. *Lashkhi A.A.* (1986) J. Algebra **99**, 1: 80-88.
9. *Fuchs L.* (1970) Infinite Abelian groups. **I**. New York-London.

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