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Mathematics

Equilibria of Point Charges on Nested Circles

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ABSTRACT. We discuss a natural problem concerned with equilibrium configurations of Coulomb potential of three positive point charges constrained to a system of nested circles in the plane. After describing our approach in general setting, several concrete problems of such type are studied in detail. First, we consider a system of three concentric circles each of which contains exactly one charge, and give a complete description of configurations, which can serve as equilibria of three positive charges. Next, we give explicit formulae for the sought charges and obtain a geometric characterization of those configurations, which can serve as stable equilibria of three positive charges. Moreover, we obtain similar results in the case of three nested circles, which are not necessarily concentric and describe the topology of the set of equilibrium configurations. Several related problems and conjectures are also presented. *© 2015 Bull. Georg. Natl. Acad. Sci.*

Keywords: *point charge, Coulomb potential, critical point, equilibrium configuration, stable equilibrium, nested circles, tangent vector, concurrent lines*

1. Stable configurations of point charges with Coulomb interaction subject to certain constraints were studied in many settings (see, e.g., [1-3]). This often reduces to investigation of critical points of Coulomb potential restricted to a given subset. Calculating the coordinates and establishing the topological type of those critical points were recognized as an important and difficult problem [1].

A new direction of research was suggested in [4-5], which led to a number of interesting results. Several developments in this direction were presented in [6-8]. The main novelty of the approach developed in [4-5] is that it focuses on the following problem, which may be thought of as the *inverse problem of electrostatics* (IPES). To describe it more precisely, let us denote by $E_Q(P)$ the Coulomb energy of collection of n point charges $Q=(q_1,...,q_n)$ placed at n-tuple of points $P = (P_1,..., P_n)$. Each pair (Q, P) will be called a configuration of charges and denoted Q/P. The general problem we are interested in can be formulated as follows.

(IPES) For a finite configuration $P = (P_1,..., P_n)$ of points in a fixed subset X can one find collection of non-zero real numbers $Q=(q_1,...,q_n)$, interpreted as values of point charges placed at the points P_i, such that the given configuration be a critical point of Coulomb potential E_Q restricted to X?

If such charges exist they are called *stationary charges for (configuration) P in X.* If P is local minimum of E_Q, then it can be considered to be equilibrium configuration for E_Q and will be called an *equilibrium of Q/P*. Any such configuration will be called a *Coulomb equilibrium in X.* In such a case, a collection of stationary charges Q will be called a *Coulomb stabilizer* of P.

These concepts and definitions substantially depend on the chosen subset X. Some choices of X are related to classical mathematical models and problems of electrostatics. For example, if X is a simple (non-self-intersecting) smooth closed curve (contour) it can be considered as a useful mathematical model of a *thin conducting loop*. Many natural and interesting problems arise if one takes conductor X to be a smooth submanifold of Euclidean space. A natural class of such problems has been discussed in [5] in the setting of *equilibria of quadratically constrained point charges.* The IPES becomes very complicated if the number of charges is big and we do not possess reasonable results in the general case. However, IPES seems to deserve attention even in the case of few charges since the results and considerations in [6], [7] show that non-trivial and interesting issues arise already for three point charges on a closed curve and four point charges at the vertices of a quadrilateral linkage.

In the present paper, we generalize some results of [6], [7] by investigating the equilibrium configurations of three point charges placed on three disjoint circles. Similar situations have already been considered in the contexts of celestial mechanics and theory of planar vortices [2], [9]. In this context, it is especially interesting to consider equilibria which are in a certain sense stable. A relevant notion of stability of Coulomb equilibrium arises if one requires that the Coulomb potential of stationary charges has a local minimum at this configuration. In the sequel such configurations will be called *stable Coulomb equilibria.* For example, it is easy to verify that the three equal charges at the vertices of regular triangle in the unit circle T form a stable configuration on T. A bunch of results on stable equilibria of point charges can be found in [8].

We generalize the setting accepted in [5], [6] by considering the case of three charges on three disjoint circles A, B, C in the plane. As usual such a triple is called *nested* if $C, \subset B \subset A$. In particular, any three concentric circles yield a nested triple. Our first main result (Theorem 1) gives a complete description of Coulomb equilibria and stable Coulomb equilibria in the case of three concentric circles. This result relies on the study of three point charges on the unit circle $T = \partial D$ performed in [5]. Here D is the unit disc and we will always assume that $A = \partial D$ and the two other circles lie inside D. We also obtain a description of Coulomb equilibria for three charges on three non-concentric nested circles (Theorem 2) and characterize those configurations which can serve as stable equilibria of three positive charges. It should be noted that equilibria of three charges on a convex curve have been studied in [6] and we make substantial use of some constructions of [6].

We also present some results concerned with the Morse theory of Coulomb potential (Theorem 3) and the topological structure of the set of all equilibrium configurations of three positive charges on three nested circles (Theorem 4). In conclusion, we present some remarks and conjectures concerned with possible generalizations of our results.

2. We begin by describing the setting of [5] in the form adjusted to our purposes. For simplicity, we only consider charges of the same sign. Extensions to the cases, where some of the charges may have different signs, are straightforward and are omitted for the sake of brevity. So in the sequel we always assume that all charges are non-zero and positive. Recall that the *Coulomb energy* of a system of point charges $Q=(q_1,..., q_n)$ placed at the points $P_1,..., P_n$ in a subset X of the plane or three-dimensional (3d) Euclidean space (up to a constant multiple which we omit as irrelevant for our considerations) is defined as

$$E_Q(P) = \sum_{i<j} \frac{q_i q_j}{d_{ij}},$$

where d_{ij} is the distance between points P_i and P_j. As is well known, the resulting force acting on q_i in position P_i (the gradient ∇E_Q of E_Q at the point P_i) is equal to

$$F_i = \sum F_{ji} = \frac{q_i q_j}{d_{ij}^3}(p_i - p_j),$$

where p_i denotes the radius-vector of point P_i and $F_{ji} = \frac{q_i q_j}{d_{ij}^3}(p_i - p_j)$ is the electrostatic force (under our assumption it is repelling) acting on q_i at P_i due to its interaction with q_j at P_j.

If charges $Q=(q_1, \dots, q_n)$ placed at $P_1, \dots, P_n$ stay in rest in X then we say that configuration $P = (P_1, \dots, P_n)$ is a *E_Q-equilibrium or Coulomb equilibrium* for the collection of charges Q, which is equivalent to requiring that P is a constrained critical point of E_Q. In such a case we say that collection of charges Q is *stationary* for P. Such configurations will be called *Coulomb equilibria* in X. Our main aim is to investigate and geometrically characterize the Coulomb equilibria in the case, where the conductor X consists of three nested circles and the number of charges equals to three. We proceed by discussing the latter setting in some detail.

3. Consider a conductor consisting of three disjoint nested circles A, B, C introduced above. We are going to consider three point charges p, q, r and assume that each of the three circles contains exactly one of the charges in the given order. The configuration space Z_3(A,B,C) defined as the set of all positions of such triples is naturally homeomorphic to the three-torus T^3.

We are now ready to formulate and prove the first main result of this note. Assume that A, B, C are concentric with the center at the origin of the plane and radii a>b>c. Let us say that a system of three points on these circles is *non-degenerate* if there is no diameter of the outer circle containing exactly two of the given points. Such a triple of points is called *balanced* if no closed half-plane contains all of them.

Theorem 1. Any non-degenerate triple of points on three concentric circles is a Coulomb equilibrium of non-zero point charges. In this case the stationary charges are uniquely defined up to multiplication by a non-zero real number and can be expressed by explicit formulae. A non-degenerate triple of points is a stable Coulomb equilibrium if and only if it is balanced.

Proof of Theorem 1. The proof is based on the explicit formulae for the stationary charges similar to those given in [6] and [8]. Denote by q_1, q_2, q_3 the sought stationary charges. Following the general strategy of [5] we aim at obtaining a system of linear equations for q_1, q_2, q_3. To this end we write down the analytic expression of the fact that each point is in equilibrium for this system of charges. By Lagrange rule, at an equilibrium the resultant force should be orthogonal to the tangent vector T_i to the corresponding circle at each point P_i, which gives three relations: $(F_i, T_i) = 0$. We consider these relations as a system of three linear equations for three variables q_j and examine its matrix. Since

$$F_{ji} = \frac{q_i q_j}{d_{ij}^3}(p_i - p_j),$$

where $p_i = \overrightarrow{OP_i}$ and $d_{ij} = \| p_i - p_j \|$, one easily verifies that the matrix of this system has the form:

$\begin{pmatrix} 0 & a_{12} & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & a_{32} & 0 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = 0,$ where $a_{ij} = \dfrac{(p_i - p_j, T_i)}{d_{ij}^3}$. Since we deal with the circles we can put $T_i = (-y_i, x_i)$. Then we get $a_{ij} = \dfrac{q_i q_j A_{ij}}{d_{ij}^3}$, where $A_{ij} = x_i y_j - x_j y_i$ is (two times) the oriented area of the triangle ΔOP_iP_j.

Let p_i be three points on different concentric circles. Then distance between p_i and p_j, $i, j = 1,2,3$ in polar coordinates is equal to $d_{ij}{}^2 = r_i^2 + r_j^2 - r_i r_j \cos(\varphi_i - \varphi_j)$.

Moreover, $T_i = (-r_i \sin\varphi_i, r_i \cos\varphi_i)$, $p_i - p_j = (r_i \cos\varphi_i - r_j \cos\varphi_j, r_i \sin\varphi_i - r_j \sin\varphi_j)$ and so

$$< p_i - p_j, T_i >= r_i r_j \sin(\varphi_i - \varphi_j).$$

Thus for the sought charges we get a system of linear equations

$$\begin{pmatrix} 0 & r_1 r_2 \sin(\varphi_1 - \varphi_2) d_{31}^3 & r_1 r_3 \sin(\varphi_1 - \varphi_3) d_{21}^3 \\ r_1 r_2 \sin(\varphi_2 - \varphi_1) d_{32}^3 & 0 & r_2 r_3 \sin(\varphi_2 - \varphi_3) d_{12}^3 \\ r_1 r_3 \sin(\varphi_3 - \varphi_1) d_{23}^3 & r_2 r_2 \sin(\varphi_3 - \varphi_2) d_{13}^3 & 0 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = 0 .$$

The matrix of this system is

$$A = \begin{pmatrix} 0 & r_1 r_2 \sin(\varphi_1 - \varphi_2) d_{31}^3 & r_1 r_3 \sin(\varphi_1 - \varphi_3) d_{21}^3 \\ r_1 r_2 \sin(\varphi_2 - \varphi_1) d_{32}^3 & 0 & r_2 r_3 \sin(\varphi_2 - \varphi_3) d_{12}^3 \\ r_1 r_3 \sin(\varphi_3 - \varphi_1) d_{23}^3 & r_2 r_2 \sin(\varphi_3 - \varphi_2) d_{13}^3 & 0 \end{pmatrix}.$$

It is now easy to see that Det A=0. Therefore this system has nontrivial solutions. Moreover, rank A = 2 if and only if there exists a pair of polar angles such that $\varphi_i \neq \varphi_j$. It follows that for any non-degenerate triple the rank of this matrix is two, so stationary charges are defined up to a constant multiple. It is also easy to see that condition that rank A = 1 is equivalent to the condition rank A = 0, which happens if and only if all three polar angles are equal.

By examining the signs of the above expressions in terms of the introduced angles one easily verifies that the three critical charges are positive if and only if the origin (centre of the circles) is inside the triangle $\Delta P_1P_2P_3$, which exactly means that the triple is balanced. To establish stability one can use the general formula on the index of a constrained critical point given in [10]. The coefficients of the bordered Hessian of E_Q can be found by a straightforward computation, which is rather long and therefore omitted. Substituting the above values of stationary charges, calculating the signs of the principal minors of the bordered Hessian and referring to the main result of [10], one verifies that the Hessian of the restriction E_Q is positive definite for balanced non-degenerate triples, which completes the proof.

In the non-concentric case, it is easy to find non-degenerate triples for which there do not exist non-trivial stationary charges. In fact, in this case the description of Coulomb equilibria is more delicate and interesting. Let us say that a triple of points from X(A,B,C) is *coherent* if the normals to the corresponding circles at these points are concurrent, i.e., have a common point. This concept is similar to that of the tripod introduced in [6] in the case of three charges on a convex curve. It turns out that one can use a geometric argument based on Ceva's theorem in a way analogous to the proof of Theorem 3 in [6] and verify that coherence yields a criterion for the existence of stationary charges. In this way one gets an analog of Theorem 1 in non-concentric case.

Theorem 2. Any coherent triple of points on three nested non-concentric circles is a Coulomb equilibrium of non-zero point charges. If the centers of circles A, B, C do not lie on the same straight line, the stationary charges are uniquely defined up to multiplication by a non-zero real number and can be expressed by explicit formulae. In the latter case a coherent non-degenerate triple of points is a stable Coulomb equilibrium if and only if the intersection point of the three normals lies inside the triangle defined by the triple.

The proof is similar to that of Theorem 1 and therefore omitted. We only add that in this case the stationary charges can also be calculated by explicit formulae involving only geometric characteristics of the given configuration. Namely, using some elementary trigonometry one finds that, for a coherent triple, the stationary charges exist and are given by the formulae

$$q_1 = -\frac{d_{12}^2 \sin b_1}{a_{12}^2 \sin a_1}, q_2 = \frac{d_{21}^2 \sin b_2}{a_{23}^2 \sin a_2}, q_3 = -\frac{d_{31}^2 \sin b_3}{a_{32}^2 \sin a_3},$$

where a_i, b_i are the angles between the i-th inner normal and two adjacent sides of triangle $\Delta P_1P_2P_3$. All remaining statements of Theorem 2 can be proved by analyzing the signs of these expressions. Moreover, using standard analytic geometry one can rewrite the coherence conditions as an explicit algebraic equation on the Cartesian coordinates of the three points. The latter equation appears useful for proving the results on the topology of coherent triples given in Section 5.

4. By a way of analogy with some papers on the relative equilibria of planar vortices (see, e.g., [9]) one may wish to investigate the *shapes* of possible equilibria of the three charges in our setting. This means that we should identify those triples of points which form congruent triangles with the same orientation, i.e. which can be transformed into each other by an orientation preserving isometry of the plane. It is then natural to introduce the *moduli space* (or *shape space*) M_3(A,B,C) of such configurations as the factor of the configuration space Z_3(A,B,C) over the group of orientation preserving isometries of the plane. It turns out that the topology of moduli space depends on the mutual location of circles A, B, C.

Proposition 1. For concentric circles, the moduli space is homeomorpic to two-torus T^2. For three non-concentric circles the moduli space is homeomorphic to T^3.

In other words, in non-concentric case there is no pair of congruent triangles in Z_3(A,B,C) having the same orientation. In the concentric case, there is a natural diagonal action of T on Z_3(A,B,C) defined by rotating all three points by the same angle $\tau \in T$. It is clear that the factor-space of this action is homeomorphic to T^2. The Coulomb potential E_Q is invariant with respect to the natural diagonal action of T. Thus, for any triple of nested circles, E_Q defines a function on the moduli space M_3(A,B,C).

Theorem 3. For any triple of nested circles and triple of positive charges, Coulomb potential E_Q is a Morse function on the moduli space M_3(A,B,C).

This follows from the exact formulae for the Hessian of E_Q mentioned in the above proofs.

Proposition 2. For three concentric circles, Coulomb potential is a Morse function. For an open set of triples (*a,b,c*), Coulomb potential is an exact Morse function, i.e. has exactly 4 critical points.

Proposition 3. For three non-concentric nested circles, Coulomb potential has not less than 8 critical points on M_3(A,B,C). For an open set of triples (a,b,c), Coulomb potential is an exact Morse function, i.e. has exactly 8 critical points, only one of which is a minimum.

Both corollaries follow from general statements of Morse theory. It is instructive to consider the case, where all charges are equal. It is easy to identify all critical configurations in this case. These results can be considered as certain analogs of Morse theory for relative equilibria of planar vortices developed in [9].

5. Having in mind applications to Coulomb control of constrained variable charges in the spirit of [6] one

may wish to investigate the topological structure of Coulomb equilibria in our setting. An obvious necessary condition for the existence of complete Coulomb control in the sense of [6] is the connectedness of the set of Coulomb equilibria. This suggests that one should try to obtain some information on the topology of this set. Denote by *E(A,B,C)* the set of all Coulomb equilibria of three point charges on three nested circles A, B, C.

Theorem 4. For a triple of concentric circles *A, B, C,* the set *E(A,B,C)* of Coulomb triples is connected. To prove this notice first that the complement of Coulomb triples in M_3(A,B,C) is contained in the union of two simple loops in T^2 intersecting at two points. More precisely, this complement consists exactly of those triples of points where exactly two of the points belong to the same diameter of A. Fixing a point on A and considering the polar angles φ, χ of the two remaining points as the coordinates on $M_3(A,B,C) = T^2$ we conclude that this complement consists of the points where $\varphi = \chi$ or $\varphi = \chi + \pi$, except the two points where $\varphi = 0$ or $\varphi = \pi$. Indeed, in the latter two configurations all three points belong to the same (horizontal) diameter of A and such configurations are Coulomb equilibria. Thus, the set E(A,B,C) consists of two simple loops intersecting at two points with the latter two points removed. It is now easy to see that this set is connected.

In fact, given two points in *E(A,B,C)* one can explicitly describe a path in *E(A,B,C)* connecting these two points. Moreover, it is now clear that the set *E(A,B,C)* is homotopic to the union of two circles with two common points. In other words, for concentric circles we have quite detailed description of the topology of Coulomb triples. Moreover, our considerations can be used to show that the set of stable Coulomb triples is not connected.

For non-concentric nested circles, an explicit description of E (A,B,C) in terms of polar angles becomes rather complicated and we cannot prove that this set is connected in general. However if the three centers of circles A, B, C lie on the same line, an easy modification of the above reasoning shows that E(A,B,C) is connected. Our conjecture is that the set of Coulomb triples is connected for any triple of nested circles.

These results and observations suggest that it should be possible to describe an explicit algorithm for Coulomb control of Coulomb triples in the spirit of [6]. However, we do not have explicit formulae for the values of charges realizing a path connecting two Coulomb triples.

6. In conclusion, we mention some possible generalizations and research perspectives. The most obvious generalization arises if one considers Coulomb equilibria of $N > 3$ point charges, where each charge is confined to one of N nested circles. There is good evidence that, for odd N and concentric circles, one has an analog of Theorem 1. For even N, situation is much more complicated as follows from the results on equilibria of four point charges on a fixed circle presented in [7]. However, we believe that generically Coulomb potential is a Morse function for any system of nested circles. It would be interesting to describe configurations corresponding to all critical point of Coulomb potential and give explicit formulae for their Morse indices in geometric terms.

Another natural possibility is to study Coulomb equilibria on other systems of nested closed curves. As a first step one could consider the case of three nested ellipses. It is very likely that an analog of Theorem 2 is true in this case.

Finally, for establishing the existence of complete Coulomb control of equilibria in our setting, it is important to know that each balanced triple yields the global minimum of Coulomb potential of stationary charges. Using the formulae given in the proof of Theorem 1, one can show that this is true for concentric circles but more general situations remain practically unexplored.

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სტატიაში განხილულია ბუნებრივი შებრუნებული ამოცანა სიბრტყეზე სამ არაგადამკვეთ წრეწირზე მდებარე დადებითი წერტილოვანი მუხტების წონასწორული მდგომარეობების შესახებ კულონურ პოტენციალურ ველში. ამოცანა დასმულია ზოგად შემთხვევაში და დეტალურად შესწავლილია რამდენიმე კონკრეტული შემთხვევა. პირველ რიგში, განხილულია სამი კონცენტრული წრეწირისაგან შედგენილი სისტემა, თითოეულზე მოთავსებულია ერთი დადებითი წერტილოვანი მუხტი და აღწერილია კონფიგურაციები, რომლებიც არიან ამ სისტემის წონასწორული მდგომარეობები. მოყვანილია ანალიზური გამოსახულება ასეთი მდგომარეობებისათვის და მიღებულია მდგრადი კონფიგურაციების გეომეტრიული დახასიათებასამი დადებითი მუხტისათვის. გარდა ამისა, აღწერილია ერთმანეთში ჩადგმული არააუცილებლად კონცენტრულ წრეწირებზემდებარე წერტილოვანი მუხტების წონასწორული მდგომარეობების ტოპოლოგია. განხილულია აგრეთვე ამოცანის მომიჯნავე პრობლემები.

REFERENCES

1. *Webb J.* (1986) Nature. 323:211-215.
2. *Exner P.* (2005) J. Phys. A: Math. Gen. A38:4795-4802.
3. *Gabrielov A., Novikov D., Shapiro B.* (2007) J. Lond. Math. Soc., 95: 443-472.
4. *Khimshiashvili G.* (2012) Bull. Georg. Natl. Acad. Sci., 6, 2: 17-22.
5. *Khimshiashvili G.* (2013) Bull. Georg. Natl. Acad. Sci., 7, 2: 15-19.
6. *Khimshiashvili G., Panina G., Siersma D.* (2015) J. Geom. Phys., 98, 2:110-117.
7. *Giorgadze G., Khimshiashvili G.* (2015) Bull. Georgian Natl. Acad. Sci., 9, 2:
8. *Giorgadze G., Khimshiashvili G.* (2015) DokladyAkad. Nauk. 465, 3:1-5.
9. *Palmore J.* (1982) Proc. Natl. Acad. Sci. USA, 79:716-718.
10. *Hassell C., Rees E.* (1993) Amer. Math. Monthly, **100**, 8:772-778.

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